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Articles 3

VOTING MODELS INCORPORATING INTERACTIONS BETWEEN VOTERS

Robert M. May and Brian Martin★

In the numerous data and observations concerning voting behaviour, there is an interactive effect that has long been noted. Namely, the results of a vote often tend to be more unanimous than would be expected on the basis of variables which otherwise predict or explain the outcome. Such a tendency for the majority's vote to snowball is variously known as the "bandwagon" or "breakage" effect. Anyone who regularly attends committee meetings should be familiar with the typical situation in which the vote tends to be unanimous in spite of everyone's indifference to the outcome. Similar behaviour, usually reduced in the magnitude of percentage change, has been documented for formal elections (Rossi and Cutright (1961), pp. 109-110; Berelson, Lazarsfeld, and McPhee (1954), p. 289).

The bandwagon effect is correctly ascribed to the interaction between different voters, the intentions of one voter inclining others to vote in the same way. As yet no models of voting behaviour which include such interactions have been posited, perhaps because no description of the form which such an interaction would take has been readily available, and perhaps because of the mathematical complexity which results from including an interaction. In any case, theories of voting which include no interaction between voters do not contain the conceptual

*Robert May is Professor of Biology, Princeton University and Brian Martin is at the School of Physics, University of Sidney. The authors would like to thank Professor H. Messel, Director of the Science Foundation within the University of Sydney, for supporting this work, and Jos Beunen and George Vorlicek for reading the manuscript.

basis to explain the bandwagon effect. In the literature, this limitation is often spelled out in the form of axioms or cautionary notes – see for example Riker and Ordeshook (1968), p. 30.

Now in a real life situation, the form of the interaction between voters is no doubt immensely complicated. An individual voter may be greatly swayed by the voting intentions of a few close friends, to a lesser degree by others; or he may be influenced by party strength or opinion polls, and so forth. A large set of such individual voting interactions, with the realistic assumption that they are all different, would be unwieldy to analyse even if ever determined. Hence in investigating the effect of including an interaction between voters we use very simple models. For each model we calculate the end result of a series of hypothetical votes. In each vote of the series, each voter is inclined to vote in agreement with the majority choice on the previous vote of certain other voters, the inclination being the greater the closer that majority was to unanimity.

In our models the actual effect of the voting interaction depends upon a suitably defined strength of interaction. For a series of votes with a small value of the strength parameter, there is a corresponding slight tendency for a majority to receive extra votes, as would be expeced, and for a series of votes undertaken with a strong interaction between voters, the eventual outcome is correspondingly close to unanimity. The important qualitative feature, however, is a critical strength of interaction: a series of votes with an interaction greater than this value tends to give a limiting result strikingly close to unanimity, compared to a series of votes with a somewhat smaller strength of interaction which gives a limiting result with a comparatively modest majority.

To determine whether this result depends sensitively on the assumptions used to obtain the model, we use several models with suitably altered assumptions. It is found that the critical strength of interaction is present whenever the interactions are mutual; that the extent of the dramatic change in the expected voting outcome near the critical interaction strength depends on the numbers of voters initially indifferent to the outcome; and that the qualitative feature of the critical interaction strength does not depend on other details of the interaction, such as the functional form used or the distribution of interaction links. The detailed models used to obtain these conclusions are presented in section I, and the results discussed in section II.

I. Models With Interactions Between Voters

(a) Interactions with all other voters: Consider a community of N voters, voting between two alternatives, Aye (denoted henceforth by "+") and Nay (denoted by "-"). Suppose that in the absence of any interaction between voters, each individual is indifferent to the voting outcome, in that each will vote + with probability 1/2, – with probability 1/2: that is

$$P(+) = \tfrac{1}{2}$$

$$P(-) = \tfrac{1}{2} \quad . \qquad (1)$$

Obviously the average result of the vote will be 50% for Aye. Due to the fact that each vote is made at random, the outcome will sometimes be less than 50% for Aye and sometimes more, with outcomes near 50% being most likely.

Now suppose a second vote is taken. If voters are not influenced by the results of the first vote, then the expected outcome will be the same, an average result of 50-50 with the same probabilities of outcomes deviating from this average. If, on the other hand, voters are influenced by the results of the first vote then this will not be true. If each voter is influenced to vote the same as the majority choice on the first vote, then an outcome near 50-50 on the second vote will be less likely. The second outcome is instead likely to be further from 50-50 than the first outcome, since all voters will have a preference one way or the other (unless the outcome of the first vote happened to be *exactly* 50-50). Next consider a third vote, fourth vote, and so on, in which for each vote each voter is influenced to vote according to the majority choice of the previous vote. If the strength of the influence in each case is larger, the larger is the majority in the previous vote, then outcomes near 50-50 will become less and less likely. After a sufficiently large number of votes a particular majority (for either Aye or Nay) will become most likely for each succeeding vote.

We now make some arbitrary assumptions concerning the nature of the interaction between the voters, and show how the expected outcome after many votes may be obtained. The sensitivity of the results obtained to the arbitrary assumptions will be studied later. Define the "fractional plurality" f to be

$$f = \frac{N(+) - N(-)}{N(+) + N(-)} , \tag{2}$$

where N(+) is the number of votes cast for Aye and N(-) the number for Nay on a particular vote, with N(+) + N(-) = N, the number of voters. Assume that each individual is influenced by the result of this particular vote, such that on the next vote he votes according to the following probabilities:

$$P(+) = \frac{e^{df}}{e^{df} + e^{-df}} ,$$
$$P(-) = \frac{e^{-df}}{e^{df} + e^{-df}} , \tag{3}$$

where d is a parameter that measures the degree of influence. On the next vote the expected number of Aye votes will be P(+)N and Nay votes P(-)N. After a large number of votes, the average fractional plurality will remain the same (although there will be fluctuations about this average). We first consider the case with $N >> 1$, so that the relative size of the fluctuations about the average fractional plurality may be neglected. If this average value of f remains the same after a vote then mathematically

$$f \equiv \frac{N(+) - N(-)}{N(+) + N(-)} = \frac{P(+)\,N - P(-)\,N}{N}$$

$$= P(+) - P(-) = \frac{e^{df} - e^{-df}}{e^{df} + e^{-df}} \equiv \tanh df, \quad (4)$$

i.e. $f = \tanh df.$

For a given constant interaction strength d, the solution to (4) for f gives the expected average fractional plurality after a large number of votes. The total vote for Aye is of course

$$\%\ \text{vote for Aye} = (100)\ N(+)\ /N = \tfrac{1}{2}(1+f)\ 100\ . \quad (5)$$

This outcome, calculated from (4), is displayed in figure 1.

Note the character of this result, as emphasized in the introduction. For no interaction, the vote is 50-50. When the interaction is small, $d < 1$, this exact 50-50 split is unaltered. But when the interaction is even only slightly greater than the critical value $d = 1$, then the individually indifferent voters reach a significant degree of unanimity for Aye, or (equally likely) for Nay. When the interaction is substantially larger than the critical value then the vote is essentially 100-0 or 0-100. It is the discontinuity in the expected voting outcome that is the striking feature. (This model is analogous to the "Bragg-Williams" theory of ferromagnetism, or of order-disorder transitions in alloys. For a much more detailed discussion of the model, see for example Huang (1963), pp. 336-341. The trivial solution $f = 0$ is of course always a mathematical solution to (4). But when $d > 1$ and the non-trivial $f \neq 0$ solution exists, as shown in Huang the $f = 0$ solution corresponds to the least likely voting outcome. Numerical solutions to (4) are most easily found by assuming a value for the product df, calculating tanh df thus giving f, and dividing the original df by the result f to get d; the product df must be suitably varied to get solutions for a range of values of d.)

The above results are for $N \gg 1$, for which fluctuations from the expected results may be neglected. For small N the deviations from the results in figure 1 will be larger. In appendix A, we display the detailed working of this model for a community comprising only 9 voters: this number of voters is already large enough for us to see how figure 1 comes about when $N \gg 1$.

(b) Interaction with four specific other voters: To see that the nature of the result in figure 1 is not dependent on the interaction being with every other voter, we consider a model in which each voter interacts only with four other voters. We assume that each voter's perference on a given vote is affected by the votes previously cast by four other voters specific to the given voter. The voter's new preference probabilities are given by (3), where f now refers to the fractional plurality of the four specific voters only. The N voters are assumed to be arranged in a 2-dimensional array, with the four specific voters being those immediately adjacent in both directions. The solution for the average fractional plurality f after a large number of votes for $N >> 1$ may be taken from the solution to the analogous problem in mathematical physics (Yang (1952), equation (96), with our d equal to Yang's 2H):

$$f = 0 \qquad , \text{ if } d < d_c$$
$$f = \pm\left[1 - (\sinh d)^{-4}\right]^{1/8}, \text{ if } d > d_c. \tag{6}$$

The critical interaction parameter, d_c, is defined by

$$d_c = \sinh^{-1}(1) = 0.8814\ldots \tag{6a}$$

The percentage vote, calculated from (5) and (6), is displayed in figure 2.

We note that the two models – interaction with all other voters and interaction with four other voters – lead to surprisingly similar results. These results differ only in the degree of unanimity presented for votes with interactions slightly greater than the critical interaction strength (closer to unanimity in figure 2) and in the particular value of the critical interaction strength (larger in figure 1, in so far as the magnitudes of the interaction parameters are comparable.)

(This model, with each voter interacting with four specified other voters, is analogous to the "2-dimensional Ising model". The form of the results in figure 2 is not sensitive to the particular choice of voter interactions: see the extensive literature on such models surveyed for example in Green and Hurst (1964).)

(c) The effect of initial preferences: Thus far we have assumed that each voter is initially indifferent to the outcome. If the different voters have a range of initial preferences, then generalized models may be developed to incorporate this fact. For example we might assume that the i-th voter, in the absence of interaction, will vote according to the probabilities

$$P(+) = \frac{e^{\varepsilon_i}}{e^{\varepsilon_i} + e^{-\varepsilon_i}} \quad ,$$
$$P(-) = \frac{e^{-\varepsilon_i}}{e^{\varepsilon_i} + e^{-\varepsilon_i}} \tag{7}$$

where the preference parameters ϵ_i are distributed in some way across the voting group. If the average vote without interactions, neglecting fluctuations, is still 50-50, we have found that such models still show a sharp discontinuity in the eventual voting outcome at a particular d. However, when the interaction is only slightly greater than the critical value, a result as close to unanimity as in figures 1 or 2 is not as likely.

This result is expected. The sharp discontinuity in the voting outcome arises when voters initially indifferent to the outcome are influenced by each other to vote for the same choice. The direction of the choice is determined by fluctuations in the early voting outcomes. When there is a spread in initial preferences, only a certain fraction of the voters are approximately indifferent to the outcome. When the interaction strength is larger than the critical value, these initially indifferent voters will tend to vote the same way. But other voters, with a strong initial preference counter to the majority choice, will tend to vote with the majority only when the interaction strength is significantly larger than the critical value.

Now assume that the voters' initial preferences are such that the expected result of the first vote is some majority for Aye or Nay. For example, ϵ_i in (7) might be the same non-zero value for each voter i. In such cases there is no sharp discontinuity in the eventual voting outcome at a particular interaction strength. However, the character of the transition persists: for example a group that votes 55-45 without interaction, might after a series of votes, be nearly unanimous if the interaction strength is greater than say d = 1.

This result is again expected, when we remember that the change in voting outcome is mainly due to the effect of mutual interactions on those voters who initially are approximately indifferent to the outcome. When most of the voters have a strong initial preference for one alternative, and few voters are initially indifferent, there will be no discontinuity in the eventual voting outcome.

Of course in many realistic voting situations many of the voters have strong initial preferences. for example they may tend to vote along party lines. Therefore in such cases an interaction between voters will affect mainly those who are undecided or indifferent, the "swinging" voters. In other situations, for example on many questions before committees, no one really cares about the outcome, and interactions rather than initial preferences decide nearly everyone's vote.

In summary, the presence of an abrupt transition in expected eventual voting outcomes will depend on at least a fraction of the voters being approximately indifferent to the outcome, and the numerical size of the transisition will depend on the size of this fraction.

(d) Non-mutual interactions: Thus far we have assumed that voters are interchangeable, that is that each voter is affected by other voters in an identical manner. For example in the model in I(b) a given voter influences four specific other voters, and each of these four influences the given voter to an identical degree. We may instead assume that certain voters have a greater degree of influence than others. This tends to reduce the extent of the discontinuity in voting outcomes near the critical interaction strength. The discontinuity results from interactions that are mutual, in which for any two voters, each influences the other to a similar degree.

To illustrate this, consider a model in which each voter is influenced to vote according to the vote of a single voter, the leader: the interaction is completely non-mutual. Let the voting preferences be given by (3), with $f = 1$ (i.e., the leader voted Aye on the previous vote). The average result of the following vote will be

$$f = \tanh d. \tag{8}$$

There is nothing reminiscent of a critical interaction strength in this outcome. (This model is analogous to the phenomenon of paramagnetism, in which electron spins line up in accordance with an external magnetic field.)

Opinion polls may at times act as a general influence on voters somewhat like a leader: many voters may be influenced by the polls, but the polls are relatively insensitive to the voters' preferences, considering the selectivity of the samples used and the time delay required for processing and publication of results. If the opinion poll is incorrect, its influence may change the result of an election. For an analysis of the bandwagon effect in relation to opinion polls, see Simon (1956), pp. 79-87.

In summary, the degree to which the change in the voting outcome for different interaction strengths is abrupt critically depends on the degree to which the interactions between voters are mutual, that is of similar strength and extent. A general influence will create a bandwagon effect which increases smoothly as the degree of influence increases, rather than abruptly as when interactions are mutual.

(e) The mathematical form of the interaction: We have assumed that the voters interact in a way such that the probability of any voter voting Aye or Nay is given by (3), where f is the fractional plurality from the previous vote of either the entire group or of four other voters, or of only the leader. This functional form is analogous to interactions in the theories of ferro-magnetism, order-disorder transitions in alloys, and gas-liquid transitions. The particular qualitative results found concerning voting, illustrated in figures 1 and 2, are analogous to the interaction of electron spins to produce the familiar magnet. However the functional forms used for the interaction are certainly not meant to be taken literally. Therefore we do not draw any analogies between physical variables,such as electron spin, and social or psychological variables, such as social class. Furthermore we do not claim that the forms for the interaction may be derived from postulates about voter psychology or other aspects of the voting process. Rather we claim that any likely functional form used for the interaction will give results qualitatively like those presented here. We have used the probabilities (3) because the Gibbs probability measure used to obtain them is quite general, and has extremely diversified applications. Its generality is confirmed by Spitzer (1971), who shows that a Gibbs random field and a Markov random field are equivalent.

To illustrate the consequences of using a different form of the probabilities, we consider here the following instead of (3):

$$P(+) = \tfrac{1}{2} + \tfrac{1}{2}fd, \tag{9}$$

$$P(-) = \tfrac{1}{2} - \tfrac{1}{2}fd.$$

The larger the value of d (up to a maximum value here of 1), the greater the inclination of each voter to vote with the previous majority. The surprising consequence of the interaction (9) is that the transition of figures 1 and 2 is made sharper. If as in I(a) each voter interacts with each other voter, (4) becomes $f = df$. For $d < 1$ the only solution is $f = 0$; for $d = 1$, the solution is $f = +1$ or $f = -1$. This latter result occurs because once the vote is unanimous, it will remain so for all following votes, and by random fluctuations a unanimous vote will occur eventually. Identical results are obtained with this form of the interaction when each voter interacts with four other voters or with only the leader.

Naturally a realistic model of the form of the interaction will be complicated in many ways: the strength of the interaction between any two voters will depend on the particular voters concerned, and vary with the voting issue at hand, the time, the voting outcome, and other factors. But given that a mutual interaction exists between voters, then unless its form is highly peculiar we expect the same qualitative results to be obtained as with the regular form of interaction (3) used here.

Finally we note that the character of the results does not depend on using d as the parameter against which to plot the voting outcome. If some non-pathological function of d is used instead, such as e^d or a positive power of d, then in the revised versions of figures 1 and 2 there will still be near unanimity when the new parameter is large, and a striking change in the expected result when the interaction strength is greater than a certain critical value.

(f) Order of the interactions: In realistic situations a long sequence of votes is unlikely to occur, even the 5 to 10 votes (see appendix A) required to closely approximate the solutions in figures 1 and 2. Instead, interactions are more likely to occur through the medium of voter preferences than through the medium of actual votes. A given voter may find out the voting preference of another voter, and be influenced by this preference, and in turn influence the preferences of others. The regular series of votes in the model may be easily interpreted as a regular series of preference interactions, with a vote which expresses the preferences occurring after certain number of these preference interactions.

The time over which mutual interactions of this sort occur may vary considerably, depending on the situation: over weeks or months in the case of a national election, and perhaps over a few seconds in the raising of hands in a small committee. Of course the interactions do not occur in neat bundles between each vote as in our model, but instead follow a complicated temporal pattern. However, the precise order in which the interactions occur is immaterial to the qualitative features of the results obtained here. This is illustrated by the model in appendix B, in which individual voters cast votes sequentially, being influenced by votes cast previously.

Removing the restriction of a rigid order in which the interactions occur removes an anomaly in the model results. If the interaction strength parameter d is negative, the earlier models give results identical to the positive values of d.

However this solution represents the unrealistic situation in which, for example when d is very negative, one vote is overwhelmingly for Aye, the next vote overwhelmingly for Nay, and so forth. This flipping of the voting outcome will not be present if the order of interactions is less uniform, and in the latter more realistic situations the likely effect of negative mutual interactions is to make the result much closer to 50-50. (This is to be distinguished from the effect of a negative *non*-mutual interaction; for an analysis of the "underdog" effect in relation to opinion polls, see again Simon (1957), pp. 79-87.)

II. Discussion

From our models of the voting process, we conclude that in a community of voters the presence of a positive interaction gives rise to a bandwagon effect. This is not surprising. Less expected is the result that if the interactions between voters are mutual, and there are a fraction of the voters approximately indifferent to the outcome, then there will exist a critical interaction strength: when the interaction strength is less than this there will be little effect due to the interaction, while when it is even only slightly larger the result will eventually be significantly closer to unanimity. The specific models we use to obtain these results illustrate a method to be used when incorporating voter interactions in a model of the voting process.

In the different parts of section I we have considered the sensitivity of these results to some of the assumptions in the model. First, the results do not depend significantly on the number of other voters with whom each voter interacts: similar results are obtained when each voter interacts with all other voters, or with four specific other voters. Second, the size of the discontinuity in voting outcomes roughly depends on the number of voters approximately indifferent to the outcome: for example when voters have an initial spread of preferences, the discontinuity is smoothed out. Third, the presence of the discontinuity itself depends on the mutuality of the interactions: when everyone follows a single voter, there only remains a bandwagon effect, depending smoothly on the strength of the interaction. Fourth, and perhaps surprisingly, the qualitative features of the voting outcomes resulting from the interaction do not depend significantly on the functional form of the interaction: rather they arise from the relative strength and distribution of the interaction links between the voters. Finally, the results do not depend strongly on the assumed regularities in the model: for example on the fact that a series of votes are taken, or that interactions occur only as a result of votes cast.

After this consideration of the numerous assumptions in the model, we hardly need emphasize that the models as presented here cannot be expected to directly apply to actual voting situations. (For some rather defensive, but highly cogent, comments on the uses of unrealistically simple models, see Brams and O'Leary (1970), p. 455, and references given there.) Our object in this paper has been, given that mutual interactions between voters exist, to draw what qualitative conclusions are possible without knowing the details of the interaction. By using a general, simple, and regular form of interaction not based on any theory specific to the voter or the voting process, we have abstracted from a realistic situation essentially only the fact that an interaction exists. Yet without knowing the details of the interaction we may hope to obtain suggestive results about the voting system, in the spirit of H. A. Simon's division between internal and external environment in *The Sciences of the Artificial.*

Appendix A

The results in section I are for a large number of voters, $N >> 1$. For small N fluctuations about these results will be significant. Here we will illustrate the extent of these fluctuations by considering a system of 9 voters, using the model of I(a). This system will also show how quickly, during the sequence of votes, the transition to the final outcome takes place.

As in I(a), each of the 9 voters interacts with all other voters according to the preference probabilities (3). On the first vote, the outcome is completely due to chance:

$$F_1(i) = \binom{N}{i} (\tfrac{1}{2})^N . \tag{10}$$

$F_n(i)$ indicates the probability that i voters will vote Aye on the n-th vote; $\binom{N}{i}$ is the number of combinations of N things taken i at a time. After the first vote, the voters will be inclined to vote the same way as the majority on the first vote, with the probabilities given by (3), where the fractional plurality is given by $f = [i - (N - i)] / N = [2i - N] / N$. Let $A(i,j)$ represent the probability that after a vote with i for Aye and N - i for Nay, a vote with j for Aye and N - j for Nay occurs. Then

$$\begin{aligned} A(i,j) &= \binom{N}{j} \left[\frac{e^{fd}}{e^{fd} + e^{-fd}}\right]^j \left[\frac{e^{-fd}}{e^{fd} + e^{-fd}}\right]^{N-j} \\ &= \binom{N}{j} \left[\frac{e^{fgd}}{e^{fd} + e^{-fd}}\right]^N , \end{aligned} \tag{11}$$

where $g = \frac{(2j - N)}{N}$ is the fractional plurality in the second vote. The outcome probabilities $F_2(j)$ on the second vote may be found from

$$F_{n+1}(j) = \sum_{i=1}^{N} F_n(i)\, A(i,j), \quad j = 1,\ldots,N, \tag{A-1}$$

with $n = 1$. Outcome probabilities for succeeding votes may be found by repeated use of the algorithm (A-1). In table 1 we present a sample of such outcome probabilities for a system of 9 voters with $d = 1.4$. (Round-off errors may cause the sum of the probabilities to be fractionally above or below 1.00 in this and in later tables.)

Table 1: Probabilities of the possible voting outcomes with 9 voters, with each voter interacting with every other voter with the interaction strength parameter d = 1.4, for selected votes. Each voter is, in isolation, equally likely to vote Aye or Nay.

Net vote for Aye, N(+) - N(-)	Number n of vote							
	1	2	3	4	5	7	10	∞
±9	.00	.03	.07	.11	.14	.17	.18	.19
±7	.02	.07	.11	.13	.14	.15	.16	.16
±5	.07	.11	.11	.10	.10	.09	.08	.08
±3	.16	.14	.11	.08	.07	.05	.04	.04
±1	.25	.15	.10	.08	.06	.04	.03	.03

After a large number of votes the outcome probabilities remain the same. The limiting probabilities may be found by setting $F_{n+1}(j) = F_n(j) \equiv F(j)$ in (A-1):

$$F(j) - \sum_{i=1}^{N} F(i)\, A(i,j) = 0, \quad j = 1,\ldots,N. \qquad \text{(A-2)}$$

The non-trivial (i.e. non-zero) solution to (A-2) is normalized by requiring $\sum_{j=1}^{N} F(j) = 1$. This solution is presented in table 1 in the column indicated "∞". It is seen that the limiting probabilities are approximately attained after a reasonably small number of votes, say 5 to 10.

In table 2 we present solutions to (A-2) for a range of interaction strengths.

Table 2: Limiting probabilities of the possible voting outcomes with 9 voters, with each voter interacting with every other voter, for a range of values of the interaction strength parameter, d. Each voter is, in isolation, equally likely to vote Aye or Nay.

Net vote for Aye, N(+) - N(-)	Value of interaction parameter d								
	0	0.4	0.6	0.8	1.0	1.2	1.4	1.6	2.0
±9	.00	.00	.01	.02	.04	.10	.19	.28	.40
±7	.02	.02	.04	.06	.10	.14	.16	.15	.08
±5	.07	.08	.09	.11	.12	.11	.08	.05	.01
±3	.16	.16	.16	.15	.12	.08	.04	.02	.00
±1	.25	.23	.20	.17	.12	.07	.03	.01	.00
Most probable outcome (Mode)	±1	±1	±1	±1	±1, ±3, or ±5	±7	±9	±9	±9

With d = 0, we simply have the distribution of voting outcomes for the sum of 9 voters each of whom is equally likely to vote Aye or Nay. For d less than about 1.0, this form of distribution persists, with an even division (±1, or maybe ±3) most likely, and unanimity (±9) very improbable. At a critical value of d around 1.0 to 1.2, each outcome from even division (±1) through to unanimity (±9) has a significant chance of occurring. When the interaction is greater than this, a unanimous (±9) or nearly unanimous (±7) vote becomes overwhelmingly probable.

The relationship between table 2 for 9 voters, and figure 1 for $N >> 1$ voters is clear. When we consider larger and larger numbers of voters, the fluctuations about the mode, or most probable voting outcome, become relatively less and less important. That is, for $N >> 1$ the pattern exhibited in table 2 is sharpened up: for $d < 1$, the vote is evenly divided (with a spread of only $N^{-1/2}$ about the 50-50 point); for $d > 1$ the vote is doubly-peaked at $\pm f$ as shown in figure 1 (with again the spread about these peaks being of width $N^{-1/2}$).

We could repeat this type of example by showing the outcome for a small number of voters each of whom interacts with four specific other voters, as in I(b). The relation between the small sample (e.g. N = 9) and that given in the main text ($N >> 1$) is precisely similar to that above.

Appendix B

In certain situations, as in the roll call of the U.S. Congress, votes are made sequentially. If each voter is influenced by the votes of the previous voters, the problem of determining the expected outcome when there is a large number of voters is in general not easily amenable to solution. However, by making some reasonable simplifying assumptions, we may obtain a model which has important characteristics of a more realistic one. In particular we may assume that a given voter in a sequential vote is influenced mostly by actual votes already cast, and comparatively little by the intentions of voters who have not yet voted.

Now consider a model, like those in section I, in which each voter has no intrinsic preference for either of the two alternatives, but in which the individual voters make their votes sequentially. Assume that for a given voter, the relative probability that he will vote + is increased by the factor exp (d/ (i-1)) for each previous + vote, and reduced by exp (-d/ (i-1)) for each previous - vote: (i-1) is the total number of previous voters in the case of the i-th voter. Thus the first voter will be indifferent (no interaction), the second voter will vote the same as the first with probability $e^d/(e^d+e^{-d})$, and so forth. For example, the probability of the sequence + - + + + - of 6 votes is

$$\left(\frac{1}{2}\right)\left(\frac{e^{-d}}{e^{d}+e^{-d}}\right)\left(\frac{1}{2}\right)\left(\frac{e^{1/3d}}{e^{1/3d}+e^{-1/3d}}\right)\left(\frac{e^{2/4d}}{e^{2/4d}+e^{-2/4d}}\right)\left(\frac{e^{-3/5d}}{e^{-3/5d}+e^{3/5d}}\right)$$

In general there will be nearly 2^{N-1} different probabilities to evaluate and classify (the first vote is arbitrary). This has been done on a computer for various values of d for from 2 to 14 voters. The results are similar to those for the cases of interaction with all other voters and interaction with four specific other voters.

For comparison with the data in appendix A, we tabulate the probabiliies for the various voting outcomes for different d, in the case of 9 voters (N=9).

Table 3: Probabilities of the possible voting outcomes with 9 voters, for a range of values of the sequential voting interaction strength parameters, d. Each voter is, in isolation, equally likely to vote Aye or Nay.

Net vote for Aye N(+) - N(-)	Value of interaction parameter d										
	0	0.2	0.4	0.5	0.6	0.8	1.0	1.2	1.4	1.8	2.2
±9	.00	.01	.03	.04	.06	.11	.18	.25	.31	.40	.45
±7	.02	.04	.07	.09	.11	.13	.14	.13	.11	.07	.04
±5	.07	.10	.11	.12	.12	.10	.08	.06	.04	.02	.01
±3	.16	.16	.14	.13	.11	.08	.05	.03	.02	.01	-
±1	.25	.20	.15	.13	.11	.07	.04	.02	.01	-	-
Most probable outcome (Mode)	±1	±1	±1	±1 or ±3	±5	±7	±9	±9	±9	±9	±9

A pattern similar to that in table 2 is here revealed. The change from more or less equal distribution (with ±1 most likely) to a near unanimous vote (±9 most likely) is not so dramatic as in the other models. This is as would be expected, since the interactions are not fully mutual: voters are not influenced by the votes cast after their own vote. However, comments in appendix A concerning fluctuations about the mode apply here with equal force, and the transition becomes sharper as the number of voters increases. A basic difference hre is that the solution for $N \rightarrow \infty$ is not known analytically, although it could be approximated numerically. The important point to be noted is that the qualitative behavior observed in this sequential voting model is the same as that of the other models which involve interations between voters.

Figure 1: The percentage vote for Aye is a very large community of voters each of whom is, in isolation, equally likely to vote Aye or Nay. This is the result after a large number of votes, in which on a given vote each voter is influenced equally by each of the votes cast on the preceding vote. The strength of the interaction is measured by d. Note that when d is small, the 50-50 vote expected for d = 0 is still most likely, but that when d is even somewhat greater than 1 the community is nearly unanimous for Aye, or (equally likely) for Nay.

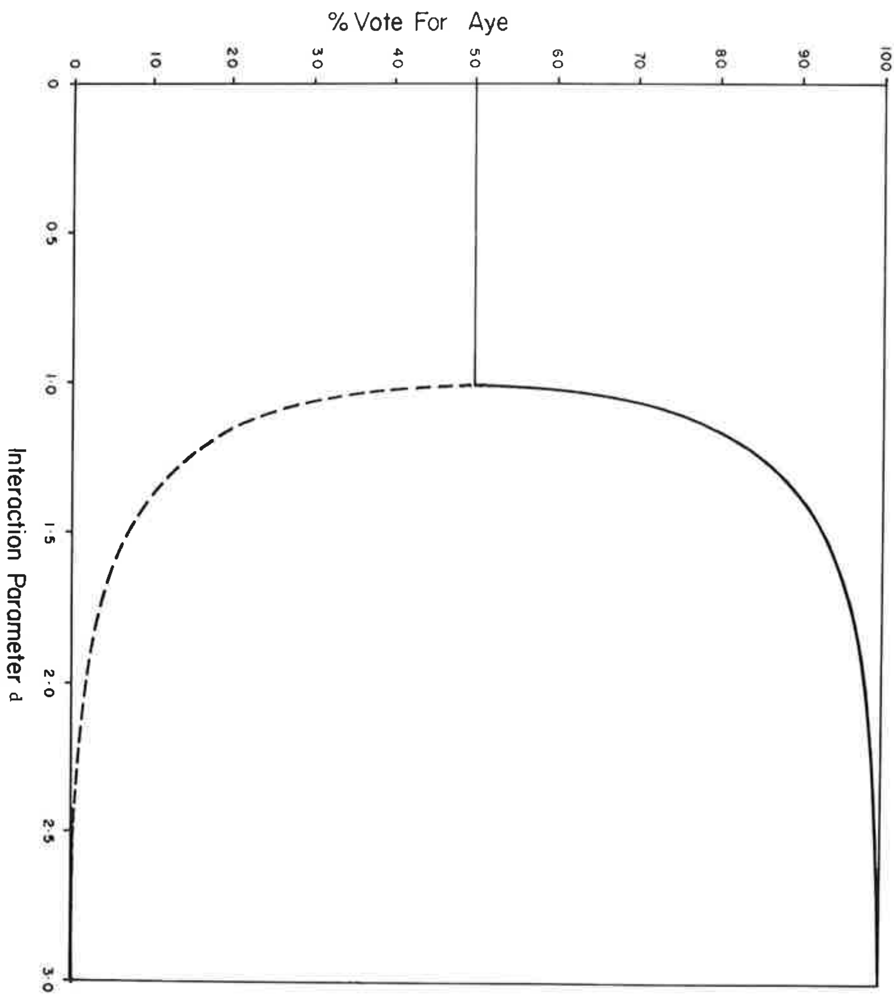
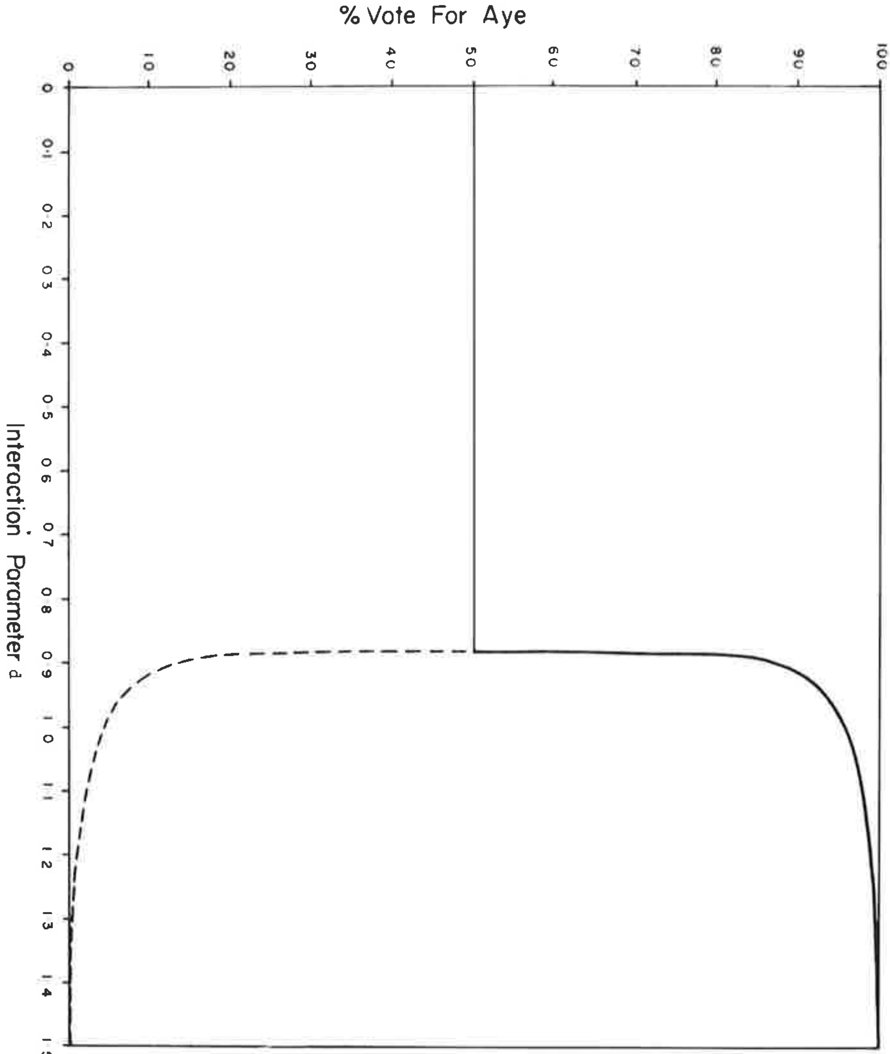


Figure 2: As for figure 1, except now each voter's opinion is influenced equally by each of four specific other voters. Note that the character of the result is the same as in figure 1, even though the type of interaction is quite different.



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