Critical Evaluation of Residence Times Calculated Using the Exponential Approximation

BRIAN MARTIN[1]

School of Physics, Sydney University, Sydney, New South Wales, Australia

The applicability and accuracy of the approximation of exponential decay in calculating residence times and transfer times are investigated by using simple models. Residence times so calculated are usually accurate to within tens of percent, while interhemispheric mixing times differ from corresponding transfer times by about a factor of 2.

A common problem in the study of turbulent fluids is to determine the average amount of time it takes a particle to move from one place to another. The residence time is a common example: it is the average time that it takes for a particle to move out of a certain region.

Often in determining such times, use is made of what will be called here the 'exponential approximation.' The total mass of a tracer particle is assumed to decrease exponentially with time in a certain region. With this assumption and with the measured change of tracer mass with time the average time for particular types of movement may be calculated. For example, this technique has been used to measure residence and other times in the atmosphere [*Peirson and Cambray*, 1967; *Fabian et al.*, 1968; *Nydal*, 1968; *Peirson*, 1969]. In this paper the applicability and accuracy of the exponential approximation will be investigated with the aid of some simple analytical and numerical models.

Although the models used here are simple, this does not mean that the results obtained do not apply to realistic situations. In particular, if the exponential approximation gives inaccurate results for a simple model, then it cannot be expected to be accurate when it is applied to a complex situation. At the very least, further investigation would be called for.

It should be noted that the exponential approximation seems to be used less and less, at least for atmospheric problems. For example, one-dimensional transport may be treated by using an altitude dependent diffusion coefficient. In addition, while the exponential approximation is used when the masses in a system are changing with time, residence times also may be determined from steady state mass distributions. Some comments on such calculations are given by *Martin* [1976].

We analyze first the application of the exponential approximation to the calculation of residence times in a one-dimensional system, and then its application to the calculation of transfer times (or times for interhemispheric mixing).

Residence Times

Consider a one-dimensional system with an absorbing barrier at $z = 0$, unbounded as $z \to \infty$, in which the density varies as $\exp(-z/H)$, where H is the scale height. Let the diffusion coefficient be a constant value K, so that the concentration c of a trace substance is determined by the diffusion equation

$$\frac{\partial c}{\partial t} = K \frac{\partial^2 c}{\partial z^2} + \frac{K}{H} \frac{\partial c}{\partial z} \tag{1}$$

Such a system might represent an idealized atmosphere, with $z = 0$ corresponding to the surface of the earth, or represent the stratosphere, with $z = 0$ corresponding to the tropopause (below which rainout occurs rapidly).

If a tracer particle is released at height $z = z_1$, the probability density $u(t, z)$ for the position z of the particle at time t is

$$u(t, z) = (4\pi Kt)^{-1/2}\left[\exp\left(-\frac{(z - z_1 + Kt/H)^2}{4Kt}\right) - \exp\left(+\frac{z_1}{H} - \frac{(z + z_1 + Kt/H)^2}{4Kt}\right)\right] \tag{2}$$

(All the formulas in this paper either are presented by *Feller* [1968, chap. 14] or may be calculated by using the methods described therein.) The probability density $p(t)$ for the time t before absorption at $z = 0$ is

$$p(t) = K \left.\frac{\partial u}{\partial z}\right|_{z=0} = \frac{z_1}{(4\pi Kt^3)^{1/2}} \exp\left(-\frac{(z_1 - Kt/H)^2}{4Kt}\right) \tag{3}$$

The average amount of time $T(z_1)$ that the particle spends before reaching $z = 0$ is

$$T(z_1) = \int_0^\infty tp(t)\, dt = \frac{z_1 H}{K} \tag{4}$$

If the tracer is initially distributed at a constant mole fraction, the average amount of time it spends before reaching $z = 0$ is $\bar{T} = H^2/K$. Thus a representative height in the system may be taken to be $z = H$.

Suppose we wish to use the exponential approximation to determine $T(z_1)$ or $\bar{T}$. Starting at a particular time $t = 0$, we measure the total tracer mass as a function of time, $m(t)$. For a particle starting at z_1 this mass is given by $m(t) = m(0)\int_0^\infty u(t, z)\, dz$. Assuming that the mass in the system drops off roughly exponentially with time, we fit the observed mass $m(t)$ to a curve

$$m(t) = m(0) \exp[-t/T_e(t, z_1)] \tag{5}$$

T_e is the residence time as determined by using the exponential approximation. In other words, at any given time t the observed mass $m(t)$ is used to calculate a residence time T_e, which thus depends on t, since $m(t)$ does not drop off in a uniform exponential manner. It is also possible to assume that the decrease proceeds from a time t_1:

$$m(t) = m(t_1) \exp[-(t - t_1)/T_e(t, z_1, t_1)] \tag{6}$$

This may be the case in many practical problems where, for

[1] Now at Centre for Resource and Environmental Studies, Australian National University, Canberra, A.C.T., Australia.

example, measurements only begin some time after various injections of mass into the system have been made.

To see roughly how accurate the exponential approximation is in these circumstances, we here compare T_e with T. It would also be possible to compare $m(t)$ with $m_e(t) = m(0) \exp(-t/T)$. We prefer the former method because normally when the exponential approximation is used, T is not known, and T_e is the result of the calculation. If desired, however, it is simple to calculate $m_e(t)/m(t) = \exp(-t(1 - T/T_e)/T)$.

In Figure 1 the ratio $T_e(t,H)/T(H)$ is plotted against the time t in units of $T(H)$. For example, if $H = 6$ km and $K = 36$ km^2 yr^{-1} ($\simeq 1.1$ m^2 s^{-1}), then $T(H) = 1$ yr. It may be noted that the ratio is for short times much greater than 1, when few particles have had time to reach $z = 0$; it then drops below 1 when the bulk of the particles (moving at the average velocity $-K/H$) reach $z = 0$; and it finally rises above 1 again as the straggling tail of the particle distribution is slowly absorbed.

The general shape observed in Figure 1 holds for the ratio $T_e(t, z_1)/T(z_1)$ at other heights. When $z_1 < H$, the drop to a minimum in the ratio is sharper and to a lower value, while for $z_1 > H$ the shape of the curve is smoother. In all cases $T_e(t, z_1)$ is several tens of percent larger than $T(z_1)$ at times several times greater than $T(z_1)$. Calculations also show that $\bar{T}_e(t)/\bar{T}$ behaves similarly to $T_e(t, H)/T(H)$.

If T_e is calculated from the drop in $m(t)$ from time t_1 according to (6), then for times past the minimum in Figure 1, $T_e(t, z, t_1)$ is greater than $T_e(t, z)$. For times significantly greater than t_1, $T_e(t, z, t_1) - T(t, z)$ appears to approach a constant value. For example, for $z = H$ and $t_1 = H^2/K$, this constant equals about 0.28. Residence times calculated by using the exponential approximation with an initial time $t_1 > 0$ also tend to be too large simply because they are based on the right-hand portion of Figure 1.

The results here are scaled so as to be independent of the size of the diffusion coefficient K. Note also that the effective mean velocity of the particle $-K/H$ is equivalent to a negative mean medium velocity V in a uniform density medium. If the density drops off with z and there is a mean velocity V as well, then $K/H - V$ replaces K/H in (1). In this case the size of K does affect the results.

A more complicated situation arises when K varies in space. One case only is treated here: $K(z) = K$ for $z > z_t$, where K is a constant, and $K(z) = rK$ for $z < z_t$, where the ratio r is a constant. This situation might correspond to a troposphere with a high diffusion coefficient and a stratosphere with a lower one, with the transition point z_t corresponding to the tropopause. In such a case the ratio r might be 10–100.

A particle released at height $z_1 > z_t$ spends an average time $(z_1 - z_t)H/K$ before first reaching z_t, an average time $[1 - \exp(-z_t/H)]H^2/K$ above z_t after first reaching z_t, and an average time $z_tH/(rK) - [1 - \exp(-z_t/H)]H^2/(rK)$ below z_t. If H is different below z_t, then an appropriate value would apply in the last expression. (These expressions may be calculated by setting up appropriate boundary value problems. Namely, for the average time before first reaching z_t, z_t is taken as an absorbing barrier (see (4)); for the average time below z_t the particle begins at z_t, and z_t is taken as a reflecting barrier; and for the average time above z_t after first reaching z_t the latter time with $r = 1$ is subtracted from the residence time $T(z_t)$. The expressions are based on the assumption that the size of the diffusion movements is small in comparison to z_t. Otherwise the time spent by a particle in the system after first reaching z_t would be reduced.) If $H = 6$ km and $K = 36$ km^2 yr^{-1}, a particle released 6 km above the tropopause ($z_t \sim 2H$) would on average spend 1 year in reaching the tropopause, about 1 year in the stratosphere after first reaching the tropopause, and a few weeks or months in the troposphere.

For this situation, calculations show that when the particles are initially significantly above z_t (at a single position, say $z_t + H$, or distributed according to $\exp(-z/H)$ above z_t only), the exponential approximation gives results similar to those for the case with constant K. When the tracer particles are initially below z_t (all at, say, $\frac{1}{2}z_t$ or distributed according to $\exp(-z/H)$ for all z), then at times much greater than the actual residence time the exponential approximation gives a significant overestimate (by a factor of 3 or 4) of the residence time. This is true because some particles move above z_t and, owing to the slower rate of diffusion there, have a longer characteristic time to reach $z=0$.

Another important situation is when rainout occurs. Calculations based on the model here show that if the average time before rainout in a region is small in comparison to the residence time of particles entering the rainout region, then the rainout region may be accurately treated as an absorbing barrier. If the bulk of the particles are initially in the rainout region, then the previous comments in reference to a high K region apply.

In summary, within the context of the simple model used here the exponential approximation usually gives results accurate to within a few tens of percent when it is used to calculate

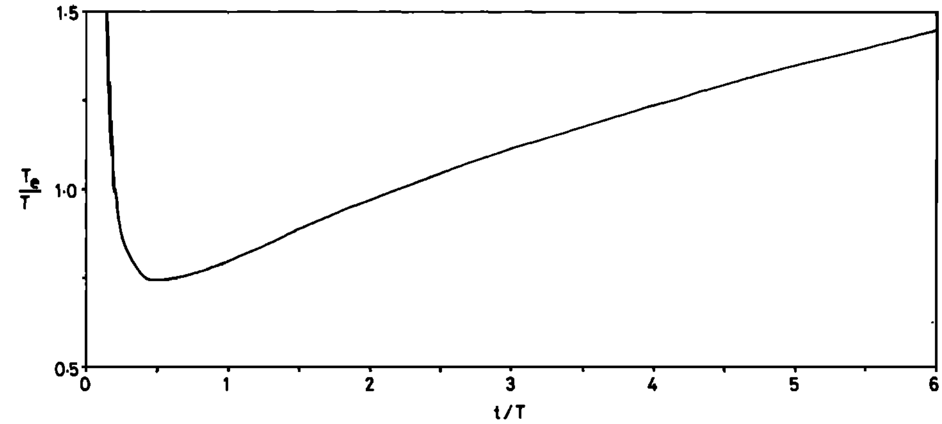

Fig. 1. The ratio $T_e(t, H)/T(H)$ (the ratio of the residence time at $z = H$ calculated by using the exponential approximation, to the actual residence time) as a function of $t/T(H)$ (the time measured in units of the actual residence time).

residence times. The most common inaccuracy will be an overestimation of the true time, especially when the initial time in the calculation is taken to be later than the time associated with the original distribution of tracer particles.

TRANSFER TIMES (AND TIMES FOR INTERHEMISPHERIC MIXING)

Consider a one-dimensional system uniform in the region $-0.5a < y < 0.5a$ with reflecting barriers at each end. Let the diffusion coefficient be a constant value K, so that the concentration c of a tracer substance is determined by

$$\partial c/\partial t = K\, \partial^2 c/\partial y^2 \tag{7}$$

Such a system might represent an idealized atmosphere, where $y = 0$ corresponds to the equator and $y = \pm 0.5a$ to the poles.

The probability density for the time t before a particle at $y = y_1$ reaches $y = 0$ is

$$q(t) = \frac{2\pi K}{a^2} \sum_{n=1}^{\infty} n \exp\left(-\frac{\pi^2 n^2 K t}{a^2}\right) \cdot \left[\sin\left(\frac{\pi n y_1}{a}\right) + \sin\left(\frac{\pi n (a - y_1)}{a}\right)\right] \tag{8}$$

Multiplying this expression by t and integrating over t from 0 to ∞ gives the average amount of time $T(y_1)$ that the particle spends before reaching $y = 0$. This is a type of residence time. If particles can also leave the system, e.g., by rainout or diffusion perpendicular to the y axis, then $T(y_1)$ is averaged over only those particles that reach $y = 0$ before leaving the system. This is a type of transfer time [*Martin*, 1976].

Suppose we wish to use the exponential approximation to determine $T(y_1)$. Starting at a particular time $t = 0$, we measure the time-varying masses $m_1(t)$ and $m_2(t)$ in the respective regions $y < 0$ and $y > 0$. The masses are assumed to change according to

$$\frac{dm_1}{dt} = \frac{m_2 - m_1}{T_e} \qquad \frac{dm_2}{dt} = \frac{m_1 - m_2}{T_e} \tag{9}$$

The value T_e that best reproduces the observed changes in mass is called an interhemispheric mixing time [e.g., *Peirson and Cambray*, 1967; *Fabian et al.*, 1968; *Peirson*, 1969]. According to (9), the interhemispheric mixing time apparently should be an approximation to the corresponding transfer time. For example, if $m_2 = 0$, say, because of rainout, then T_e will correspond to a residence time (a special case of a transfer time) for the mass m_1.

Accordingly, to compare the transfer time with the value obtained using the exponential approximation, we determine $T_e(t, y_1)$ from (for $y_1 > 0$) the solution to (9):

$$\frac{m_2(t) - m_1(t)}{m_2(t) + m_1(t)} = \frac{m_2(0) - m_1(0)}{m_2(0) + m_1(0)} \exp\left(\frac{-2t}{T_e(t, y_1)}\right) \tag{10}$$

In Figure 2, $T_e(t, 0.25a)/T(0.25a)$ is plotted as a function of t in units of $T(0.25a)$. There is a noticeable resemblance to Figure 1, with, however, a difference of a factor of about 2. Apparently T_e in this case refers to a different quantity from the average time required for a particle to move from $y = y_1$ to $y = 0$. (Note that the average time required for a particle to move from y_1 to $-y_1$ in general does not equal $2T(y_1)$, so that it is not necessarily convenient to associate T_e with particle trajectories from y_1 to $-y_1$.)

It might be argued in retrospect that there is no reason to expect a correspondence between $T_e(y_1)$ and $T(y_1)$. If this is the case, then the point to be made here is that the meaning of times calculated using the exponential approximation may not be easily interpretable from the equations, such as (9), used to obtain them.

Similar results are obtained for $T_e(t, y_1)/T(y_1)$ for $y_1 \neq 0.25a$, the sharpness and depth of the minimum being increased for small y_1. Results closely similar to those in Figure 2 are obtained if the volume associated with each position varies as $\cos(\pi y/a)$, which corresponds to the variation in atmospheric volume with latitude. The results are also similar if the initial distribution of tracer particles is proportional to the volume for, say, $y > 0$ (one hemisphere) and if there is a mean wind (say, away from $y = 0$).

Consider now the case in which particles continuously leave the system as well as diffuse in the y direction. The transfer time $T(y_1)$ will be decreased, since particles taking longer to reach $y = 0$ are more likely to be removed from the system first. On the other hand, the solution (10) is unchanged. If, for example, the average time for rainout or diffusion out of the system equals $T(y_1)$ (calculated without such removal), then $T(y_1)$ will be reduced by about a factor of 2. In this example, T_e typically would be about 4 times as large as T. It might be argued that T_e refers to a mixing time which assumes that exit from the system does not occur. This is a reasonable interpretation if the removal process is rainout or diffusion (in the z direction) which does not interact with the diffusion in the y direction. If the diffusion processes in the two directions interact (e.g., through spatially varying diffusion coefficients), then it is not obvious what T_e refers to.

In summary, interhemispheric mixing times calculated by using the exponential approximation have the same general

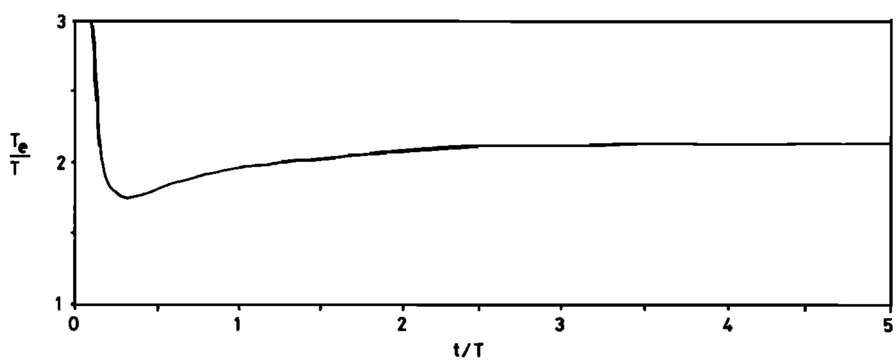

Fig. 2. The ratio $T_e(t, 0.25a)/T(0.25a)$ (the ratio of the transfer time from $y_1 = 0.25a$ to $y = 0$ calculated by using the exponential approximation, to the actual transfer time) as a function of $t/T(0.25a)$ (the time measured in units of the actual transfer time).

characteristics as residence times so calculated, except for being in addition roughly twice as large as the corresponding transfer times.

Conclusion

When residence and transfer times are calculated by using the assumption of exponential variation of mass with time, care should be taken. The calculations here show that depending on the circumstances, times so calculated may be systematically inaccurate, as in the case of residence times, and also apparently may refer to a different quality from that suggested by the equations used to calculate the times, as in the case of transfer times. The work here indicates that simple models can be used to estimate the applicability and accuracy of the exponential approximation.

Acknowledgments. I wish to thank Robert G. Hewitt for helpful conversations and the reviewers of the manuscript for useful comments. Support for this work was received from H. Messel and the Science Foundation for Physics at Sydney University.

References

Fabian, P., W. F. Libby, and C. E. Palmer, Stratospheric residence time and interhemispheric mixing of strontium 90 from fallout in rain, *J. Geophys. Res., 73,* 3611–3616, 1968.

Feller, W., *An Introduction to Probability Theory and Its Applications,* 3rd ed., vol. 1, John Wiley, New York, 1968.

Martin, B., On calculating residence and transfer times, *J. Geophys. Res., 81*(3), 387–394, 1976.

Nydal, R., Further investigation on the transfer of radiocarbon in nature, *J. Geophys. Res., 73,* 3617–3634, 1968.

Peirson, D. H., Interhemispheric transfer of radioactive pollution from nuclear explosions, *Phil. Trans. Roy. Soc. London, Ser. A, 265,* 295–300, 1969.

Peirson, D. H., and R. S. Cambray, Interhemispheric transfer of debris from nuclear explosions using a simple atmospheric model, *Nature, 216,* 755–758, 1967.

(Received July 2, 1975;
revised October 1, 1975;
accepted October 1, 1975.)