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Solutions for radiative transfer in magnetic atmospheres

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Summary. An analytical solution method is presented for the problem of radiative transfer in the presence of a magnetic field and of absorbing lines. Previously available solution methods are also presented and tested, and the usefulness and limitations of the different methods are discussed.

1 Introduction

The solution to the problem of radiative transfer in the presence of a magnetic field and of absorbing lines (giving rise to Zeeman-split lines) requires the treatment of four inter-dependent differential equations for the four Stokes parameters. This problem is important for both solar physics and astrophysics.

A number of solution methods have been proposed. Unno (1956) and Stepanov (1958) present analytical solutions for the special case in which the opacities are constant and the source function is linear in optical depth; however, other approaches must be sought for realistic atmospheres. Shipman (1971) offers a simple approximate method for calculating the flux and circular polarization. Beckers (1969) gives a full numerical solution based on the Runge–Kutta algorithm, and Hardorp, Shore & Wittmann (1976) present an approximate numerical solution based on a perturbation method.

The object of this paper is two-fold. First, we present a new method for integrating the radiative transfer equations using an explicit analytical solution. This method has been used successfully in the restricted case of three Stokes parameters for computing the theoretical spectra of magnetic white dwarfs (Martin & Wickramasinghe 1979a; Wickramasinghe & Martin 1979). Secondly, we present a comparison of this method with the above solution methods currently in use to solve the radiative transfer problem for polarized light, in terms of their accuracy, computational speed and convenience.

2 An analytical solution

We first present a solution to the radiative transfer problem for the restricted set of three Stokes parameters I, Q and V, since this restricted set is sufficient for many astrophysical problems and because the solution is considerably simpler and demonstrates the method of solution. Afterwards the solution for the full set of four Stokes parameters will be presented.

The radiative transfer equations for the restricted set of three Stokes parameters I, Q and V are (Unno 1956)

$$\mu \frac{dI}{d\tau} = \eta_I(I - B) + \eta_Q Q + \eta_V V, \tag{1}$$

$$\mu \frac{dQ}{d\tau} = \eta_Q(I - B) + \eta_I Q, \tag{2}$$

$$\mu \frac{dV}{d\tau} = \eta_V(I - B) + \eta_I V, \tag{3}$$

where

$$\eta_I = \tfrac{1}{2}\eta_p \sin^2\psi + \tfrac{1}{4}(\eta_l + \eta_r)(1 + \cos^2\psi), \tag{4}$$

$$\eta_Q = [\tfrac{1}{2}\eta_p - \tfrac{1}{4}(\eta_l + \eta_r)] \sin^2\psi, \tag{5}$$

$$\eta_V = \tfrac{1}{2}(\eta_r - \eta_l) \cos\psi. \tag{6}$$

$\mu = \cos\theta$, where θ is the angle between the propagation direction and the normal to the surface of the star, ψ is the angle between the propagation direction and the direction of the local magnetic field, B is the local source function, and η_p, η_l and η_r are the ratios of the total absorption coefficient of the three shifted Zeeman components to the (unpolarized) continuum absorption coefficient. (The 'total absorption coefficient' includes both continuous and line absorption, and hence allows for polarization in the continuum. If the continuum is not polarized, then η_p, η_l and η_r can be taken to correspond to the ratio of line-to-continuum opacities, and η_I in our formulation is replaced by $1 + \eta_I$, as is done in most presentations of the radiative transfer equations.) The optical depth τ scale refers to unpolarized light in the continuum, i.e. $d\tau = -\kappa_p\, dz$ where κ_p is the absorption coefficient for p-electrons in the continuum.

To solve the equations, we take a series of optical depths $\tau_0 = 0, \tau_1, \tau_2, \ldots, \tau_{\max}$. At each optical depth τ_i there is an Unno solution for the radiation emerging from the sphere whose physical radius corresponds to that optical depth:

$$I_U = B\left(1 + \frac{\beta\mu\eta_I}{\eta_I^2 - \eta_Q^2 - \eta_V^2}\right), \tag{7}$$

$$Q_U = -B\frac{\beta\mu\eta_Q}{\eta_I^2 - \eta_Q^2 - \eta_V^2}, \tag{8}$$

$$V_U = -B\frac{\beta\mu\eta_V}{\eta_I^2 - \eta_Q^2 - \eta_V^2}, \tag{9}$$

where B, $\beta \equiv B^{-1}dB/d\tau$, η_I, η_Q and η_V are evaluated at τ_i. For the initial solution at $\tau_{\max}$ we adopt the expressions (7)–(9) evaluated at $\tau_{\max}$.

Given a solution I_n, Q_n and V_n at any optical depth τ_n, we calculate the solution at the next lowest depth τ_m by assuming that in the region $\tau_m \leqslant \tau \leqslant \tau_n$,

$$I = I_a + I_b\tau + \sum_{i=1}^{3} I_{c_i} \exp(a_i\tau), \tag{10}$$

$$Q = Q_a + Q_b\tau + \sum_{i=1}^{3} Q_{c_i} \exp(a_i\tau), \tag{11}$$

$$V = V_a + V_b\tau + \sum_{i=1}^{3} V_{c_i} \exp(a_i\tau), \tag{12}$$

$$B = B_a + \gamma\tau. \tag{13}$$

Substituting equations (10)–(13) into (1)–(3) and equating constant terms and terms in τ and $\exp(a_i\tau)$, values for the unknown parameters in equations (10)–(13) can be obtained. The solution at τ_m is

$$I_m = I_U + i_1 + i_2, \tag{14}$$

$$Q_m = Q_U + q_1 + \eta_Q q_2, \tag{15}$$

$$V_m = V_U - \eta_Q q_1/\eta_V + \eta_V q_2, \tag{16}$$

where I_U, Q_U and V_U are given by equations (7)–(9) evaluated at τ_m, and

$$i_1 = \tfrac{1}{2}(t_1 + t_2 - \gamma/a_1) \exp(-a_1\Delta\tau), \tag{17}$$

$$i_2 = \tfrac{1}{2}(t_1 - t_2 - \gamma/a_2) \exp(-a_2\Delta\tau), \tag{18}$$

$$q_1 = \eta_V(\eta_Q^2 + \eta_V^2)^{-1}(\eta_V Q_n - \eta_Q V_n) \exp(-\eta_I\Delta\tau/\mu), \tag{19}$$

$$q_2 = (i_1 - i_2)/(\eta_Q^2 + \eta_V^2)^{1/2}, \tag{20}$$

$$t_1 = I_n - B_n, \tag{21}$$

$$t_2 = (\eta_Q Q_n + \eta_V V_n)/(\eta_Q^2 + \eta_V^2)^{1/2}, \tag{22}$$

$$a_1 = (\eta_I + (\eta_Q^2 + \eta_V^2)^{1/2})/\mu, \tag{23}$$

$$a_2 = (\eta_I - (\eta_Q^2 + \eta_V^2)^{1/2})/\mu, \tag{24}$$

$$= (\tfrac{1}{2}\eta_p(\eta_r + \eta_l) \sin^2\psi + \eta_l\eta_r \cos^2\psi)/(a_1\mu^2), \tag{25}$$

$$\Delta\tau = \tau_n - \tau_m. \tag{26}$$

To obtain maximum accuracy, the parameters γ, η_I, η_Q and η_V in the above expressions should be evaluated at $\tfrac{1}{2}(\tau_n + \tau_m)$. An alternative using values only at the τ_i is to set $\gamma = (B_m - B_n)/(\tau_m - \tau_n)$ and take the values of η_I, η_Q and η_V as the average of their values at τ_m and τ_n. The expressions (14)–(26) are given in a form suitable for automatic computation. In some special cases the round-off error in calculating a_2 by equation (24) is severe, a problem which the expression (25) avoids. Note that if the initial solution at τ_{max} is an Unno solution as specified by equations (7) to (9), then q_1 will always vanish.

The assumption made in deriving the solution (14)–(26) is that the source function B is linear in τ between τ_m and τ_n, and that η_I, η_Q and η_V are constant in the same region. If a sufficient number of τ values are adopted, this assumption should be closely satisfied. We find that with constant opacities and a grey atmosphere, that three points are sufficient to give accuracy within 1 per cent; with a more realistic temperature structure, six points are sufficient to give 1 per cent accuracy. With non-constant opacities, more integration points usually would be required for a similar accuracy.

For the full set of four Stokes parameters I, Q, U and V the radiative transfer equations may be written (Hardorp *et al.* 1976)

$$\mu \frac{dI}{d\tau} = \eta_I(I - B) + \eta_Q Q + \eta_V V, \tag{27}$$

$$\mu \frac{dQ}{d\tau} = \eta_Q (I - B) + \eta_I Q - \rho_R U, \tag{28}$$

$$\mu \frac{dU}{d\tau} = \rho_R Q + \eta_I U - \rho_W V, \tag{29}$$

$$\mu \frac{dV}{d\tau} = \eta_V (I - B) + \rho_W U + \eta_I V. \tag{30}$$

The pair $\binom{Q}{U}$ of solutions must be multiplied by

$$\begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{pmatrix},$$

where ϕ is the azimuth with respect to an arbitrary x-axis at right angles to the line of sight. Hardorp *et al.* (1976) also note that if $d\phi/d\tau \neq 0$ then the solution to equations (27)–(30) is obtained by replacing ρ_R by $\rho_R - 2\mu(d\phi/d\tau)$. The terms ρ_R and ρ_W introduce anomalous dispersion into the system.

The Unno solution for equations (27)–(30) may be written

$$I_U = B + \frac{\mu B \beta}{\eta_I - \eta_V^2/\eta_{IW} - \eta_{Q\rho}^2/\eta_{IR}}, \tag{31}$$

$$Q_U = -\frac{\eta_{Q\rho}}{\eta_{IR}} (I_U - B), \tag{32}$$

$$V_U = \left(-\frac{\eta_V}{\eta_{IW}} - \frac{\rho \eta_{Q\rho}}{\eta_{IR}}\right) (I_U - B), \tag{33}$$

$$U_U = (\rho_W V_U - \rho_R Q_U)/\eta_I, \tag{34}$$

where

$$\eta_{IW} = \eta_I + \rho_W^2/\eta_I. \tag{35}$$

$$\rho = \rho_R \rho_W/(\eta_I \eta_{IW}), \tag{36}$$

$$\eta_{IR} = \eta_I + \rho_R^2/\eta_I - \rho^2 \eta_{IW}, \tag{37}$$

$$\eta_{Q\rho} = \eta_Q + \rho \eta_V. \tag{38}$$

This Unno solution (31)–(38) is again adopted as the initial solution at $\tau_{\max}$. Given a solution at τ_n, solutions at τ_m are calculated using expressions like equations (10)–(12) but with four exponential terms. Substituting the expressions into equations (27)–(30) results in four eigenvalues a_i, two real and two complex. The solution at τ_m may be written

$$I_m = I_U + u_1(x_1 - x_2) + z_1 h_1, \tag{39}$$

$$Q_m = Q_U + u_2(x_1 + x_2) + z_2 h_2, \tag{40}$$

$$U_m = U_U + u_3(x_1 - x_2) + z_3 h_1, \tag{41}$$

$$V_m = V_U + x_1 + x_2 + h_2, \tag{42}$$

where I_U, Q_U, U_U and V_U are given by equations (31)–(34) evaluated at τ_m, and

$$h_1 = [x_3 \sin(y\Delta\tau/\mu) + x_4 \cos(y\Delta\tau/\mu)] \exp(-\eta_I \Delta\tau/\mu), \tag{43}$$

$$h_2 = [x_3 \cos(y\Delta\tau/\mu) - x_4 \sin(y\Delta\tau/\mu)] \exp(-\eta_I \Delta\tau/\mu), \tag{44}$$

$$x_4 = (u_3 g_1 - u_1 g_3)/(u_3 z_1 - u_1 z_3), \tag{45}$$

$$x_3 = (g_2 - u_2 g_4)/(z_2 - u_2), \tag{46}$$

$$x_2 = \tfrac{1}{2}(g_4 - g_3/u_3 - x_3 + z_3 x_4/u_3) \exp(-(\eta_I - x)\Delta\tau/\mu), \tag{47}$$

$$x_1 = \tfrac{1}{2}(g_4 + g_3/u_3 - x_3 - z_3 x_4/u_3) \exp(-(\eta_I + x)\Delta\tau/\mu), \tag{48}$$

$$g_1 = I_n - (I_U + \gamma\Delta\tau), \tag{49}$$

$$g_2 = Q_n - Q_U, \tag{50}$$

$$g_3 = U_n - U_U, \tag{51}$$

$$g_4 = V_n - V_U, \tag{52}$$

where again in equations (49)–(52), I_U, Q_U, U_U and V_U are given by equations (31)–(34) evaluated at τ_m, and

$$z_3 = (\rho_W - \rho_R z_2)/y, \tag{53}$$

$$z_1 = -(\eta_V + \eta_Q z_2)/y, \tag{54}$$

$$z_2 = (\eta_Q \eta_V + \rho_R \rho_W)/(-y^2 + \rho_R^2 - \eta_Q^2), \tag{55}$$

$$u_3 = (-\rho_W + \rho_R u_2)/x, \tag{56}$$

$$u_1 = (\eta_V + \eta_Q u_2)/x, \tag{57}$$

$$u_2 = (\eta_Q \eta_V + \rho_R \rho_W)/(x^2 + \rho_R^2 - \eta_Q^2), \tag{58}$$

$$y = (b + q)^{1/2}, \tag{59}$$

$$x = (-b + q)^{1/2}, \tag{60}$$

$$q = (b^2 + c)^{1/2}, \tag{61}$$

$$b = \tfrac{1}{2}(\rho_R^2 + \rho_W^2 - \eta_Q^2 - \eta_V^2), \tag{62}$$

$$c = (\rho_W \eta_Q + \rho_R \eta_V)^2. \tag{63}$$

The solution (39)–(63) is given in a form suitable for automatic computation (calculating in reverse order). Note that if there is no anomalous dispersion ($\rho_R = \rho_W = 0$) then $y = 0$ and the solution breaks down: see equations (53) and (54). The solution (14)–(26) should be used in this case.

3 Numerical solutions

Besides the analytical solution presented for the restricted set of three Stokes parameters by Unno (1956), there are at least three proposed approaches for solving the radiative transfer problem in a magnetic field, given by Shipman (1971), Beckers (1969) and Hardorp *et al.* (1976). Tests of these methods will be presented in Section 4 after the brief description here.

The analytical approach of Unno can be applied to atmospheres in which the source function B is non-linear in τ by obtaining an approximate linear source function (Unsöld 1955, Section 41) which for our purposes may be written

$$B = B_0(1 + \beta\tau), \tag{64}$$

$$\beta = \phi/(1 - 2\phi/3), \tag{65}$$

$$\phi = (3/16)\alpha/[1 - \exp(-\alpha)], \tag{66}$$

$$\alpha = h\nu/kT_e. \tag{67}$$

The solution is then obtained using equations (7)–(9) or (31)–(34) where the opacities are evaluated at $\tau^* = 2/3$. This approach can only be expected to apply successfully to atmospheres such as a grey atmosphere which are similar to the Unno atmosphere.

Shipman's method is based on the assumption that the effect on flux and circular polarization due to increased absorption in a single Zeeman component is proportional to the change in flux in a single component system with suitably altered opacity. Here we apply the method to calculate intensity only, for the sake of comparison with the other methods. The single component equation for the intensity is

$$\mu\frac{dI}{d\tau} = \eta(I - B). \tag{68}$$

If I_c is the surface intensity of the solution to equation (68) with continuum absorption only, and I_r is the surface intensity of the solution to equation (68) in which η is replaced by η_r, the opacity of the single Zeeman component which is additionally absorbing (here arbitrarily assumed to be the component which absorbs right-handed circularly polarized light; η_r also includes the r-component of the continuum, in agreement with the notation adopted previously), then the Shipman solution is

$$I_0 = \tfrac{1}{2}(I_r + I_c), \tag{69}$$

$$V_0 = (I_r - I_c)/(I_r + I_c). \tag{70}$$

The method makes no prediction for Q_0 or U_0. Much more sophisticated looking formulae can be derived when taking into account averaging over the stellar disc and when incorporating a particular T–τ relationship (see, e.g. Landstreet & Angel 1975; Brown *et al.* 1977), but equations (69) and (70) represent the essence of the method.

Beckers' approach (see also the corrections and supplementary work of Wittmann 1972, 1974) is simply based on a Runge–Kutta approximation to the derivatives in equations (1)–(3) or (27)–(30), and step-wise numerical solution outwards beginning at an optical depth well within the atmosphere.

Hardorp *et al.*'s approach is based on an orthogonal transformation of the equations (27)–(30), and a perturbation-type solution of the resulting equations. Up to second order their solution may be written

$$I = 2^{-1/2}(J_2 + J_4), \tag{71}$$

$$Q = (\eta_Q^2 + \eta_V^2)^{-1/2}(2^{-1/2}\eta_Q(J_2 - J_4) + \eta_V J_1), \tag{72}$$

$$U = J_3, \tag{73}$$

$$V = (\eta_Q^2 + \eta_V^2)^{-1/2}(2^{-1/2}\eta_V(J_2 - J_4) - \eta_Q J_1), \tag{74}$$

$$J_1 = J_1^{(1)} + J_1^{(2)}, \tag{75}$$

$$J_2 = J_2^{(0)} + J_2^{(2)}, \tag{76}$$

$$J_3 = J_3^{(1)} + J_3^{(2)}, \tag{77}$$

$$J_4 = J_4^{(0)} + J_4^{(2)}, \tag{78}$$

$$J_2^{(0)} = 2^{-1/2} \exp(A_1) \int_\tau^\infty a_1 B \exp(-A_1)\, d\tau', \tag{79}$$

$$J_4^{(0)} = 2^{-1/2} \exp(A_2) \int_\tau^\infty a_2 B \exp(-A_2)\, d\tau', \tag{80}$$

$$A_1 = \int_0^\tau a_1\, d\tau' \tag{81}$$

$$A_2 = \int_0^\tau a_2\, d\tau', \tag{82}$$

where a_1 and a_2 are given by equations (23) and (25), and

$$J_1^{(1)} = \exp(A_3) \int_\tau^\infty D(J_2^{(0)} - J_4^{(0)}) \exp(-A_3)\, d\tau'/\mu, \tag{83}$$

$$J_3^{(1)} = -\exp(A_3) \int_\tau^\infty R_2(J_2^{(0)} - J_4^{(0)}) \exp(-A_3)\, d\tau'/\mu, \tag{84}$$

$$A_3 = \mu^{-1} \int_0^\tau \eta_I\, d\tau', \tag{85}$$

$$J_1^{(2)} = \exp(A_3) \int_\tau^\infty R_1 J_3^{(1)} \exp(-A_3)\, d\tau'/\mu, \tag{86}$$

$$J_2^{(2)} = -\exp(A_1) \int_\tau^\infty (DJ_1^{(1)} - R_2 J_3^{(1)}) \exp(-A_1)\, d\tau'/\mu, \tag{87}$$

$$J_3^{(2)} = -\exp(A_3) \int_\tau^\infty R_1 J_1^{(1)} \exp(-A_3)\, d\tau'/\mu, \tag{88}$$

$$J_4^{(2)} = \exp(A_2) \int_\tau^\infty (DJ_1^{(1)} - R_2 J_3^{(1)}) \exp(-A_2)\, d\tau'/\mu, \tag{89}$$

$$D = -2^{-1/2} \mu (\eta_Q^2 + \eta_V^2)^{-1} \left(\eta_Q \frac{d\eta_V}{d\tau} - \eta_V \frac{d\eta_Q}{d\tau} \right), \tag{90}$$

$$R_1 = (\eta_Q^2 + \eta_V^2)^{-1/2} (\rho_R \eta_V + \rho_W \eta_Q), \tag{91}$$

$$R_2 = 2^{-1/2} (\eta_Q^2 + \eta_V^2)^{-1/2} (\rho_R \eta_Q - \rho_W \eta_V). \tag{92}$$

The solution (72)–(73) for Q and U must be rotated as indicated previously for any solution to the radiative transfer equations as formulated in equations (27)–(30). In equations (79), (80), (91) and (92) we have incorporated slight corrections in Hardorp *et al.*'s formulae as written. In each of the integrals (79)–(89) the factors composing the integrand are functions of τ' although this is not explicitly noted, and each of the expressions on the left hand side of the equals signs are functions of τ. Hardorp *et al.* note that $D = 0$ unless different types of Zeeman components mix (or in the case of τ-dependent continuum polarization, which amounts to the same thing).

4 Tests of the methods

In testing the methods, 31 different τ points were used. This large number of τ points was adopted so that the results would depend solely on the solution method and not on approximations involved in quadrature; as in the case of the analytical solution, a much smaller number of points would be sufficient for practical purposes. Three different atmospheric structures were used: (a) an Unno source function, $B(\tau) = 1 + 0.2\,\tau$; (b) a grey atmosphere temperature structure $T = 12\,000(3\,\tau/4 + \frac{1}{2})^{1/4}$, $\lambda = 5000$ Å; and (c) a 'real' atmosphere adopted from Wickramasinghe (1972) with $T_e = 12\,000$ K and $\lambda = 5000$ Å (see Table 1 for the T–τ relationship in the real atmosphere).

Table 1. Temperature T as a function of optical depth τ for a model atmosphere from Wickramasinghe (1972), T_e = 12 000 K, λ = 5000 Å.

τ	T	τ	T
0	8625	1.2	13191
.001	8676	1.6	13920
.002	8727	2	14464
.004	8786	3	15452
.006	8830	4	16169
.008	8867	5	16750
.01	8900	6	17179
.015	8963	10	18686
.02	9027	14	19626
.03	9131	22	21186
.04	9222	30	22425
.07	9457		
.1	9669		
.15	9951		
.2	10233		
.3	10693		
.4	11086		
.6	11771		
.8	12330		
1	12757		

Some details of the computational procedure are in order. For Shipman's method, the intensities I_r and I_c were obtained using the analytical solution with all η's equal. This is costly in computational time but guarantees that the method comparison does not involve differences in quadrature techniques. For the Runge–Kutta method, the initial optical depth was taken as $\tau = 5$, and steps of $\Delta\tau = 10^{-3}$ were used. For Hardorp *et al.*'s method, integrals of the form

$$\int_{\tau}^{\infty} a_1 B \exp(-A_1)\, d\tau' \tag{93}$$

were transformed to the form

$$\int_{0}^{\exp(-A_1)} B(\tau')\, dx, \tag{94}$$

where τ' is related to x by

$$A_1 = \int_{0}^{\tau'} a_1\, d\tau'' = -\ln(x). \tag{95}$$

The integrals of form (94) were then evaluated by utilizing 31 x-coordinates evenly spaced in the interval $x = (0, \exp(-A_1))$. This in effect represents a quadrature formula in which the optical depths are the values $\tau'(x)$ and the weights are all unity. In both the Runge–Kutta and Hardorp *et al.*'s methods the values of B, η_I, η_Q and η_V were found by interpolating linearly in the tables giving these parameters at the 31 fixed optical depths.

For the tests, a variety of functional dependences of η_p, η_l and η_r on τ were used. In addition, different tests were made in which η_r was multiplied by a factor $1 + 10^k$, $k = -2, -1, \ldots, 6$, to simulate the effect of lines of varying strength. Some representative results are presented in Tables 2–6. Results for the real atmosphere case are emphasized

Table 2. Comparison of solutions to the radiative transfer problem in a magnetic Unno atmosphere with source function $B = 1 + 0.2\tau$, using the analytical method, the Unno solution, the method of Shipman (1971), the Runge–Kutta method and the method of Hardorp *et al.* (1976), for several sets of constant opacities, with $\mu = 0.8$, $\cos\psi = 0.7$, $\cos 2\phi = 0.6$ and $\eta_p = \eta_l = 1$.

		Analytical	Unno	Shipman	Runge-Kutta	Hardorp et al.
$\rho_R=\rho_W=0$						
$\eta_r=1$	I_0	1.16000	1.16000	1.16000	1.15982	1.15827
$\eta_r=1.1$	I_0	1.15445	1.15445	1.15273	1.15427	1.15279
	Q_0	.00190	.00190	-	.00190	.00188
	V_0	-.00521	-.00521	-.00631	-.00521	-.00516
$\eta_r=2$	I_0	1.12585	1.12585	1.12000	1.12566	1.12449
	Q_0	.01169	.01169	-	.01169	.01156
	V_0	-.03209	-.03209	-.03571	-.03209	-.03174
$\eta_r=$ 100001	I_0	1.08000	1.08000	1.08000	-	1.07914
	Q_0	.02738	.02738	-	-	.02709
	V_0	-.07517	-.07517	-.07407	-	-.07436
$\rho_R=1.5$; $\rho_W=0.75$						
$\eta_r=2$	I_0	1.12276	1.12276	1.12000	1.12256	1.11665
	Q_0	.00855	.00855	-	.00855	.01083
	U_0	-.00836	-.00836	-	-.00836	-.03552
	V_0	-.02482	-.02482	-.03571	-.02482	-.01337

Table 3. Comparison of solutions to the radiative transfer problem in a magnetic grey atmosphere, $T_e = 12\,000$ K, using the analytical method, an Unno solution with $B = 1 + 0.738\tau$, the method of Shipman (1971), the Runge–Kutta method and the method of Hardorp *et al.* (1976), for several sets of constant opacities, with $\mu = 0.8$, $\cos\psi = 0.7$, $\cos 2\phi = 0.6$ and $\eta_p = \eta_l = 1$.

		Analytical	Unno	Shipman	Runge-Kutta	Hardorp et al.
$\rho_R=\rho_W=0$						
$\eta_r=1$	I_0	1.66320	1.59032	1.66320	1.66270	1.66000
$\eta_r=1.1$	I_0	1.64481	1.56985	1.63900	1.64428	1.64174
	Q_0	.00629	.00700	-	.00630	.00625
	V_0	-.01728	-.01923	-.01477	-.01730	-.01716
$\eta_r=2$	I_0	1.54196	1.46430	1.51895	1.54135	1.53920
	Q_0	.04150	.04313	-	.04153	.04135
	V_0	-.11391	-.11840	-.09497	-.11401	-.11350
$\eta_r=$ 100001	I_0	1.33161	1.29516	1.33161	-	1.33001
	Q_0	.11350	.10103	-	-	.11295
	V_0	-.31157	-.27733	-.24902	-	-.31006
$\rho_R=1.5$; $\rho_W=0.75$						
$\eta_r=2$	I_0	1.53375	1.45292	1.51895	1.53306	1.52057
	Q_0	.03464	.03156	-	.03466	.04438
	U_0	-.02476	-.03086	-	-.02477	-.09373
	V_0	-.09212	-.09159	-.09497	-.09218	-.06515

Table 4. Comparison of solutions to the radiative transfer problem in a real atmosphere from Wickramasinghe (1972), $T_e = 12\,000$ K, in a magnetic field, using the analytical method, an Unno solution with $B = 1 + 0.738\tau$, the method of Shipman (1971), the Range–Kutta method and the method of Hardorp *et al.* (1976), for several sets of constant opacities, with $\mu = 0.8$, $\cos\psi = 0.7$, $\cos 2\phi = 0.6$ and $\eta_p = \eta_l = 1$.

		Analytical	Unno	Shipman	Runge-Kutta	Hardorp et al.
$\rho_R=\rho_W=0$						
$\eta_r=1$	I_0	2.64677	1.59032	2.64677	2.64584	2.64333
$\eta_r=1.1$	I_0	2.61165	1.56985	2.60043	2.61067	2.60840
	Q_0	.01202	.00700	-	.01204	.01196
	V_0	-.03299	-.01923	-.01782	-.03305	-.03283
$\eta_r=2$	I_0	2.40377	1.46430	2.35422	2.40256	2.40062
	Q_0	.08317	.04313	-	.08327	.08308
	V_0	-.22832	-.11840	-.12427	-.22859	-.22805
$\eta_r=$ 100001	I_0	1.82349	1.29516	1.82347	-	1.82178
	Q_0	.28179	.10103	-	-	.28120
	V_0	-.77354	-.27733	-.45150	-	-.77193
$\rho_R=1.5$; $\rho_W=0.75$						
$\eta_r=2$	I_0	2.39083	1.45292	2.35422	2.38942	2.37374
	Q_0	.07448	.03156	-	.07454	.09424
	U_0	-.04111	-.03086	-	-.04114	-.14524
	V_0	-.19042	-.09159	-.12427	-.19063	-.15209

Table 5. Comparison of solutions to the radiative transfer problem in a real atmosphere from Wickramasinghe (1972), $T_e = 12000\,K$, in a magnetic field, using the analytical method, an Unno solution, the method of Shipman (1971), the Runge–Kutta method and the method of Hardorp *et al.* (1976), for several sets of τ-dependent opacities, with $\mu = 0.8$, $\cos\psi = 0.7$, $\cos 2\phi = 0.6$ and $\eta_p = \eta_l = \eta$ with $\eta = 0.2 + \tau$.

		Analytical	Unno	Shipman	Runge-Kutta	Hardorp et al.
$\rho_R = \rho_W = 0$						
$\eta_r = \eta$	I_0	2.97477	1.68114	2.97477	2.98068	2.97358
$\eta_r = 1.1\eta$	I_0	2.94920	1.65752	2.94092	2.95500	2.94799
	Q_0	.00875	.00808	–	.00879	.00876
	V_0	-.02403	-.02219	-.01151	-.02412	-.02405
$\eta_r = 2\eta$	I_0	2.78625	1.53574	2.74365	2.79155	2.78500
	Q_0	.06453	.04977	–	.06473	.06455
	V_0	-.17713	-.13662	-.08424	-.17770	-.17719
$\eta_r = 100001\eta$	I_0	1.98794	1.34057	1.98779	–	1.98734
	Q_0	.33778	.11657	–	–	.33757
	V_0	-.92722	-.31999	-.49652	–	-.92667
$\rho_R = 1.5$; $\rho_W = 0.75$						
$\eta_r = 2\eta$	I_0	2.77945	1.52119	2.74365	2.78459	2.77536
	Q_0	.05809	.03115	–	.05787	.08945
	U_0	-.06537	-.03927	–	-.06609	-.15323
	V_0	-.13541	-.10234	-.08424	-.13551	-.10959

since this case provides the most severe test. In Table 7 a comparison of computational execution times is presented.

It will be convenient to comment on these results and on the general convenience of the methods at the same time. First, the Unno solution is only satisfactory for giving a rough approximation to an accurate solution in grey-type atmospheres. For our real atmosphere the Unno solution is quite inaccurate, not surprisingly since it depends only on the effective temperature T_e and the opacities at $\tau^* = 2/3$. To be fair, the approximation to an Unno atmosphere presented by Unsöld (1955) was designed originally for grey-type atmospheres; we have applied it to the real atmosphere case for the sake of completeness.

Second, Shipman's method usually gives a reasonable result for the intensity but for circular polarization it cannot be relied upon. For example, in the case of a real atmosphere and a weak single component line with τ-independent opacities (Table 4), Shipman's method underestimates the circular polarization by roughly 50 per cent. Analytical extensions of this method which are currently in use for the computation of continuum polarization in magnetic white dwarfs (Angel 1977; Brown *et al.* 1977) thus may lead to large errors even if a realistic atmospheric structure is adopted. Two of the extensions of Shipman's method are analysed in Martin & Wickramasinghe (1979b). The poor results achieved using Shipman's method are not surprising, remembering that the method is not based on rigorous mathematical analysis of the problem. The poor values for circular polarization seem to come about from lack of treatment of linear polarization: the Shipman values for V_0 seem to be some sort of non-linear average of the actual Q_0, U_0 and V_0 values.

Table 6. Comparison of solutions to the radiative transfer problem in a real atmosphere from Wickramasinghe (1972), $T_e = 12\,000$ K, in a magnetic field, using the analytical method, an Unno solution, the method of Shipman (1971), the Runge–Kutta method and the method of Hardorp *et al.* (1976), for several sets of τ-dependent opacities, with $\mu = 0.8$, $\cos\psi = 0.7$, $\cos 2\phi = 0.6$ and $\eta_l = 1, \eta_p = 1 + \tau$.

		Analytical	Unno	Shipman	Runge-Kutta	Hardorp et al.
$\rho_R=\rho_W=0$						
$\eta_r=1$	I_0	2.55216	1.51543	2.55216	2.55208	2.54935
	Q_0	-.09461	-.07489	-	-.09376	-.09402
	V_0	0	0	0	0	0
$\eta_r=1.1$	I_0	2.52072	1.49784	2.55313	2.52056	2.51787
	Q_0	-.07868	-.06485	-	-.07783	-.14350
	V_0	-.02684	-.01443	.00038	-.02692	-.01913
$\eta_r=2$	I_0	2.32773	1.40380	2.30691	2.32724	2.32514
	Q_0	.01701	-.01113	-	.01778	.01428
	V_0	-.19254	-.09162	-.10631	-.19306	-.19307
$\eta_r=$ 100001	I_0	1.75273	1.24032	1.77616	-	1.75139
	Q_0	.25757	.08226	-	-	.25711
	V_0	-.70706	-.22580	-.43690	-	-.70580
$\rho_R=1.5$; $\rho_W=0.75$						
$\eta_r=2$	I_0	2.32713	1.40244	2.30691	2.32646	2.32262
	Q_0	.03618	-.00332	-	.03671	.02524
	U_0	-.02393	-.03010	-	-.02388	-.04434
	V_0	-.18522	-.08382	-.10631	-.18553	-.17855

Table 7. Approximate computer execution time in seconds on a Univac 1110 for determining solutions to the radiative transfer problem in a magnetic field for 27 different temperature structure and opacity sets, using the analytical method, an Unno solution, the method of Shipman (1971), the Runge–Kutta method and the method of Hardorp *et al.* (1976).

Method	Time: 3 Stokes parameters (single precision arithmetic)	4 Stokes parameters (double precision arithmetic)
Analytical	0.22	0.61
Unno	0.05	0.08
Shipman	0.21	-
Runge-Kutta	47.6	56.2
Hardorp et al.	17.6	22.0

Third, the Runge–Kutta approach offers an accurate solution for all test cases, but is not useful in the case of deep lines because of the excessive computational time required. (Obviously, suitably small τ step sizes could have been used to obtain solutions for $\eta_r > 10^3$ – but at the cost of increasing computation time by a corresponding factor and also at the cost of increased round-off error.) Therefore, this method can be recommended only for weak lines or when the number of integrations required is quite small.

Fourth is the method of Hardorp *et al.* It is fairly expensive with computational time compared to the analytical solution method, but it is unlike the Runge–Kutta approach in that no increase in computation time is required for deeper lines. In many cases $\rho_R = \rho_W = D = 0$ and the zero order term gives an exact solution. When $\rho_R = \rho_W = 0$ but $D \neq 0$ the inclusion of up to second order terms appears to give excellent accuracy. However, the method seems inadequate on some special cases of τ-dependent η's and in the case of large values of ρ_R and ρ_W. We have found that when η_Q and η_V change sign at some optical depth – even when $D = 0$ – Hardorp *et al.*'s method gives incorrect results, as seen in Table 6. (We have not been able to discover the precise theoretical reason for the breakdown of the method in this special case.) Also note that certain η–τ relations, such as $\eta_p = 1 + \tau^{1/2}$, $\eta_l = 1$, $\eta_r = 1 + \tau$, give rise to a singularity in D at $\tau = 0$, though this may not give problems in a given numerical solution. Finally, when ρ_R and ρ_W are sizable as in the final case in Tables 2–6, the perturbation method does not converge quickly enough to give satisfactory results.

5 Conclusion

We have demonstrated an explicit analytical solution to the radiative transfer problem in Zeeman-split lines. Also we have compared the accuracy, computational speed and convenience of several other available solution methods on several test problems.

For the problem as tested, we have no hesitation in recommending the analytical solution as presented here on all three criteria. The method is particularly suitable for dealing with radiative transfer in magnetic white dwarfs where the continuum is polarized and strong lines are involved. Results obtained by using this method for computing cyclotron absorption and hydrogen line absorption in magnetic white dwarfs have been discussed by Martin & Wickramasinghe (1979a) and Wickramasinghe & Martin (1979). We would recommend though that a check be made on the correct programming of the solution by comparing results with those presented here, or by comparing them with a Runge–Kutta solution.

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